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ABSTRACT

A theorem with a number of equivalent alternatives is proposed as an extension of the classical Gordan theorem of the alternative. The theorem can handle nonzero unrestricted variables which cannot be directly treated by ordinary theorems of the alternative. Like the Gordan theorem, the extended theorem has the stability feature that small perturbations in the data will not invalidate an alternative that is in force. The theorem has useful applications in establishing the boundedness and uniqueness of feasible points of polyhedral sets and of solutions to linear programming problems.

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## SIGNIFICANCE AND EXPLANATION

In deriving optimality conditions and duality relations of optimization theory certain theorems, called theorem of the alternative, play a key role. These theorem characterize the solvability of a certain system of inequalities by the unsolvability of a related system of inequalities. We extend here one of the fundamental theorems of the alternative in such a way that it can handle certain types of variables not easily handled before. As applications we can give conditions which characterize uniqueness or boundedness of solution of linear programming problems. Elsewhere the theorem has been used by chemical engineers to give conditions under which maximum energy recovery is possible in a heat exchanger network under a certain disturbance range.

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A STABLE THEOREM OF THE ALTERNATIVE:  
AN EXTENSION OF THE GORDAN THEOREM

O. L. Mangasarian

1. Introduction

Theorems of the alternative play a key role in mathematical programming [4, 8, 2, 7]. Among the best known and very useful theorems of the alternative is the Gordan theorem [6, 4, 8] which states that for any  $m \times n$  real matrix  $D$  the following are equivalent:

(i)  $Dy > 0$  has a solution  $y$  in  $R^n$

(ii)  $D^T v = 0, 0 \neq v \geq 0$ , has no solution  $v$  in  $R^m$ .

Here  $R^n$  denotes the  $n$ -dimensional real Euclidean space and the superscript  $T$  denotes the transpose. These two alternatives however are not the only ones that can be stated for the Gordan theorem. For example it can be easily shown [5] that (i) and (ii) are also equivalent to the following

(iii) For each  $(c, h)$  in  $R^{m+n}$  the linear program

$$\max_{v \in R^m} \{c^T v \mid D^T v = h, v \geq 0\}$$

is either infeasible or has a nonempty bounded optimal solution set.

It is also elementary to verify that (i) is also equivalent to

(iv)  $Dy \geq c$  has a solution  $y$  in  $R^n$  for each  $c$  in  $R^m$

and

(v)  $\bar{D}y > 0$  has a solution  $y$  in  $R^n$  for each  $m \times n$  matrix  $\bar{D}$  such that  $\|\bar{D} - D\| \leq \delta$  for some  $\delta > 0$ .

Here  $\|\cdot\|$  denotes any matrix norm. The existence of these various equivalent alternatives prompts one to look for a general type of theorem which subsumes the Gordan alternatives. One such very general extension can be achieved by using the general regularity theory of Robinson [12, 13]. Our approach here employs the more basic framework of the classical theorems of the alternative [4, 8] to arrive at the desired extension. A key role in the extension is played by the stability alternative (v) above, namely that  $\bar{D}y > 0$  has a solution  $y$  in  $R^n$  for all  $\bar{D}$  in  $R^{m \times n}$  (the space of  $m \times n$  real matrices) that are sufficiently close to  $D$ . This alternative shows that the set of data (matrices in  $R^{m \times n}$ ) for which alternative (i) holds in an open set in  $R^{mn}$ . By contrast it can be shown by means of simple examples that the set of data satisfying either of the Farkas theorem alternatives [3, 8]:  $Ax \geq 0, b^T x < 0$  has a solution  $x$  in  $R^n$ , or  $A^T u = b, u \geq 0$  has a solution  $u$  in  $R^m$ , where  $A$  is in  $R^{m \times n}$  and  $b$  is in  $R^n$ , is not an open set in  $R^{mn+n}$ .

In view of the important role played by the stability alternative (v) we shall term our extension of the Gordan theorem, Theorem 1 below, a stable theorem of the alternative. The aptness of this terminology will be more apparent from Theorem 2 of the next section which shows that if any one of the alternatives of Theorem 1 holds then they all hold for sufficiently small perturbations of the data. In Section 3 of the paper we exhibit some applications of the stable theorem of the alternative in the form of characterizations of boundedness and uniqueness of solution of linear programs. We also mention a practical application in engineering.

In order to be concrete we give now definitions of a theorem of the alternative and a stable theorem of the alternative.

Definition 1. A theorem of the alternative is an equivalence relationship between the solvability of a system of linear equalities and inequalities and the unsolvability of a related system of linear equalities and inequalities.

The solvability of the first system and the unsolvability of the second system will be referred to as equivalent alternatives or more simply alternatives.

Definition 2. A stable theorem of the alternative is a theorem of the alternative with more than two equivalent alternatives and such that if one of its alternatives holds then it, and consequently all the other alternatives, hold for all sufficiently small but arbitrary perturbations of the (constant) data constituting the linear equalities and inequalities of the alternatives.

Some of the interesting features of stable theorems of the alternative are:

- (a) They often involve nonzero unrestricted variables that are usually not handled by ordinary theorems of the alternative. (See alternative (i') of Theorem 1 below.)
- (b) They give useful existence properties for perturbations of systems of linear inequalities and equalities. (See alternatives (i), (ii) and (iv) of Theorem 1 below.)
- (c) They give useful boundedness results for certain polyhedral sets and linear programs. (See alternatives (v') and (vii') of Theorem 1 and also Theorem 2, below.)

We briefly describe now the notation used. All matrices and vectors are real. For the  $m \times n$  matrix  $A$  we write  $A \in R^{m \times n}$  and denote row  $i$  by  $A_i$ , column  $j$  by  $A_{.j}$  and the element in row  $i$  and column  $j$  by  $A_{ij}$ .

For  $x$  in the real  $n$ -dimensional Euclidean space  $R^n$ , element  $j$  is denoted by  $x_j$ . All vectors are column vectors unless transposed by the superscript  $T$ . For  $I \subset \{1, \dots, m\}$  and  $J \subset \{1, \dots, n\}$ :  $A_I$  denotes the submatrix of  $A$  with rows  $A_i$ ,  $i \in I$ ;  $A_{\cdot J}$  denotes the submatrix of  $A$  with columns  $A_{\cdot j}$ ,  $j \in J$ ;  $A_{IJ}$  denotes the submatrix of  $A$  with elements  $A_{ij}$ ,  $i \in I$ ,  $j \in J$  and  $x_J$  denotes  $x_i$ ,  $i \in J$ .  $\|x\|$  will denote an arbitrary but fixed norm on  $R^n$  and the corresponding induced matrix norm  $\max_{\|x\|=1} \|Ax\|$  will be denoted

by  $\|A\|$ . For brevity we shall often omit mentioning the dimensionality of a vector or a matrix, it being obvious from the context. The vector  $e$  will be a vector of ones in the appropriate Euclidean space.



## 2. A stable theorem of the alternative

We begin with our principal result which subsumes the Gordan theorem and which gives many useful equivalent alternatives. Some of these equivalences can also be derived from Robinson's general regularity approach which uses multifunction theory [13] and some from Rockafellar's convex analysis results [14]. For example the equivalence between (iii), (i') and (i) of Theorem 1 below can be established after some work by using the equivalence between I, II and III respectively of Theorem 3 of [13]. Similarly the equivalence between (i') and (v') of Theorem 1 below can also be established by using Rockafellar's Theorem 8.4 [14] which states that the boundedness of a nonempty closed convex set is equivalent to its recession cone containing the origin only. In keeping with the spirit of theorems of the alternative our proofs here will rely mainly on these theorems.

Theorem 1 (A stable theorem of the alternative). Let  $A, B, C$  and  $D$  be fixed matrices in  $R^{m_1 \times n_1}$ ,  $R^{m_1 \times n_2}$ ,  $R^{m_2 \times n_1}$  and  $R^{m_2 \times n_2}$  respectively and let  $a$  and  $c$  be fixed vectors in  $R^{m_1}$  and  $R^{m_2}$  respectively. The following are equivalent:

$$(i) \quad Ax + By = a + \gamma b$$

$$Cx + Dy \geq c + \gamma d$$

$$x \geq 0$$

$$1 \geq \gamma > 0$$

has solution  $(x, y, \gamma)$

for each  $(b, d)$

$$(i') \quad A^T u + C^T v \leq 0$$

$$B^T u + D^T v = 0$$

$$a^T u + c^T v \geq 0$$

$$v \geq 0$$

$(u, v) \neq 0$

has no solution  $(u, v)$

$$(ii) \tilde{A}x + \tilde{B}y = \tilde{a}$$

$$\tilde{C}x + \tilde{D}y \geq \tilde{c}$$

$$x \geq 0$$

has solution  $(x,y)$

for each  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{a}, \tilde{c})$

such that:

$$\max\{\|\tilde{A}-A\|, \|\tilde{B}-B\|, \|\tilde{C}-C\|, \|\tilde{D}-D\|, \|\tilde{a}-a\|, \|\tilde{c}-c\|\} \leq \epsilon \text{ for some } \epsilon > 0$$

$$(iii) Ax + By = a$$

$$Cx + Dy > c$$

$$x > 0$$

has solution  $(x,y)$  and:

rows of  $[A \ B]$  are linearly independent

$$(iv) \bar{A}x + \bar{B}y = \bar{a}$$

$$\bar{C}x + \bar{D}y > \bar{c}$$

$$x > 0$$

has solution  $(x,y)$  and

rows of  $[\bar{A} \ \bar{B}]$  are linearly

independent for each  $(\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{a}, \bar{c})$

such that:

$$\max\{\|\bar{A}-A\|, \|\bar{B}-B\|, \|\bar{C}-C\|, \|\bar{D}-D\|, \|\bar{a}-a\|, \|\bar{c}-c\|\} \leq \delta \text{ for some } \delta > 0$$

$$(ii') \tilde{A}^T u + \tilde{C}^T v \leq 0$$

$$\tilde{B}^T u + \tilde{D}^T v = 0$$

$$\tilde{a}^T u + \tilde{c}^T v > 0$$

$$v \geq 0$$

has no solution  $(u,v)$

for each  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{a}, \tilde{c})$

such that:

$$(iii') \quad 0 \neq \begin{pmatrix} -A^T u - C^T v \\ a^T u + c^T v \\ v \end{pmatrix} \geq 0$$

$$B^T u + D^T v = 0$$

has no solution  $(u,v)$  and:

$$(iv') \quad 0 \neq \begin{pmatrix} -\bar{A}^T u - \bar{C}^T v \\ \bar{a}^T u + \bar{c}^T v \\ v \end{pmatrix} \geq 0$$

$$\bar{B}^T u + \bar{D}^T v = 0$$

has no solution  $(u,v)$  and

rows of  $[\bar{A} \ \bar{B}]$  are linearly

independent for each  $(\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{a}, \bar{c})$

such that:

(v') For each  $(g, h, \alpha)$  the set

$$S(g, h, \alpha) = \left\{ (u, v) \left| \begin{array}{l} A^T u + C^T v \leq g \\ B^T u + D^T v = h \\ a^T u + c^T v \geq \alpha \\ v \geq 0 \end{array} \right. \right\}$$

is empty or bounded

(vi') For some  $(g, h, \alpha)$  the set

$S(g, h, \alpha)$  of (v') is a singleton.

(vii) There exists an  $\epsilon > 0$  such that (vii') There exists an  $\epsilon > 0$  such that for

for each  $(\tilde{a}, \tilde{c})$  satisfying

$$\|(\tilde{a}, \tilde{c}) - (a, c)\| \leq \epsilon \text{ and for}$$

each  $(g, h)$  the linear program

$$\min_{x, y} \left\{ g^T x + h^T y \left| \begin{array}{l} Ax + By = \tilde{a} \\ Cx + Dy \geq \tilde{c} \\ x \geq 0 \end{array} \right. \right\}$$

is feasible and either its objective function is unbounded below or the set of dual optimal multipliers is nonempty and bounded.

each  $(\tilde{a}, \tilde{c})$  satisfying

$$\|(\tilde{a}, \tilde{c}) - (a, c)\| \leq \epsilon \text{ and for}$$

each  $(g, h)$  the linear program

$$\max_{u, v} \left\{ \tilde{a}^T u + \tilde{c}^T v \left| \begin{array}{l} A^T u + C^T v \leq g \\ B^T u + D^T v = h \\ v \geq 0 \end{array} \right. \right\}$$

is either infeasible or has a nonempty bounded optimal solution set.

(viii) There exists an  $\epsilon > 0$  and

$(g, h)$  such that for each

$(\tilde{a}, \tilde{c})$  satisfying

$$\|(\tilde{a}, \tilde{c}) - (a, c)\| \leq \epsilon \text{ the linear}$$

program of (vii) has a unique

dual optimal multiplier.

(viii') There exists an  $\epsilon > 0$  and  $(g, h)$

such that for each  $(\tilde{a}, \tilde{c})$  satisfying

$$\|(\tilde{a}, \tilde{c}) - (a, c)\| \leq \epsilon \text{ the linear}$$

program of (vii') has a unique

solution.

Proof

(i)  $\Leftrightarrow$  (i') : (i') is equivalent to

$$A^T u + C^T v \leq 0, B^T u + D^T v = 0, -a^T u - c^T v \leq 0, -v \leq 0, -r^T u - s^T v < 0$$

not having a solution  $(u,v)$  for each  $(r,s)$  in  $R^{m_1+n_1}$ . By Motzkin's theorem of the alternative [11,8] this is equivalent to

$$\bar{A}\bar{x} + \bar{B}\bar{y} - a\bar{\xi} - r = 0, \bar{C}\bar{x} + \bar{D}\bar{y} - c\bar{\xi} - s \geq 0, \bar{x} \geq 0, \bar{\xi} \geq 0$$

having a solution  $(\bar{x}, \bar{y}, \bar{\xi})$  for each  $(r,s)$  in  $R^{m_1+n_1}$ . By defining  $b = r - a$  and  $d = s - c$  this is equivalent to

$$\bar{A}\bar{x} + \bar{B}\bar{y} = a(\bar{\xi}+1) + b, \bar{C}\bar{x} + \bar{D}\bar{y} \geq c(\bar{\xi}+1) + d, \bar{x} \geq 0, \bar{\xi} \geq 0$$

having a solution  $(\bar{x}, \bar{y}, \bar{\xi})$  for each  $(b,d)$  in  $R^{m_1+n_1}$ . This is equivalent to (i) if we make the identifications  $x = \frac{\bar{x}}{1+\bar{\xi}}, y = \frac{\bar{y}}{1+\bar{\xi}}$  and  $\gamma = \frac{1}{1+\bar{\xi}} > 0$ .

(ii)  $\Leftrightarrow$  (ii'): By Motzkin's theorem.

(iii)  $\Leftrightarrow$  (iii'): By Motzkin's theorem.

(iv)  $\Leftrightarrow$  (iv'): By Motzkin's theorem.

(vii)  $\Leftrightarrow$  (vii'): By linear programming duality [1].

(viii)  $\Leftrightarrow$  (viii'): By linear programming duality.

(i')  $\Rightarrow$  (ii): If (ii) did not hold then there exists a sequence

$\{A^i, B^i, C^i, D^i, a^i, c^i\}, i = 1, 2, \dots$ , converging to  $(A, B, C, D, a, c)$  such that

for  $i = 1, 2, \dots$ ,

$$A^i x + B^i y = a^i, C^i x + D^i y \geq c^i, x \geq 0$$

has no solution  $(x,y)$ . By Motzkin's theorem this is equivalent to

$$A^iT_u + C^iT_v \leq 0, B^iT_u + D^iT_v = 0, a^iT_u + c^iT_v > 0, v \geq 0$$

having a solution  $(u^i, v^i)$  for  $i = 1, 2, \dots$ . By letting  $\bar{u}^i = \frac{u^i}{\|u^i, v^i\|}$  and

$$\bar{v}^i = \frac{v^i}{\|u^i, v^i\|}, \text{ it follows by the Bolzano-Weierstrass theorem that the bounded}$$

sequence  $\{\bar{u}^i, \bar{v}^i\}$  has an accumulative point  $(\bar{u}, \bar{v})$  satisfying

$$A^T \bar{u} + C^T \bar{v} \leq 0, B^T \bar{u} + D^T \bar{v} = 0, a^T \bar{u} + c^T \bar{v} \geq 0, \bar{v} \geq 0, \|\bar{u}, \bar{v}\| = 1$$

which contradicts (i').

(ii)  $\Rightarrow$  (iii): Since  $Ax + By = \tilde{a}$  has a solution  $(x, y)$  for each  $\tilde{a}$  such that  $\|\tilde{a} - a\| \leq \epsilon$  for some  $\epsilon > 0$ , it follows that the rows of  $[A \ B]$  are linearly independent. Setting  $\tilde{A} = A, \tilde{B} = B, \tilde{C} = C, \tilde{D} = D, \tilde{a} = a - \frac{Ae\epsilon}{k}$  and  $\tilde{c} = c - \frac{Ce\epsilon}{k} + \frac{e\epsilon}{2\|e\|}$  where  $k > 2 \max\{\|Ae\|, \|Ce\|\}$  we get from (ii) that there exists  $(\bar{x}, \bar{y})$  satisfying

$$A\bar{x} + B\bar{y} = a - \frac{Ae\epsilon}{k}, C\bar{x} + D\bar{y} \geq c - \frac{Ce\epsilon}{k} + \frac{e\epsilon}{2\|e\|}, \bar{x} \geq 0.$$

Hence it follows that  $x = \bar{x} + \frac{e\epsilon}{k} > 0$  and  $y = \bar{y}$  satisfy the conditions of (iii).

(iii')  $\Rightarrow$  (iv): Since the set of matrices with full row rank is an open set it follows that  $[\bar{A} \ \bar{B}]$  is of full row rank for sufficiently small  $\delta > 0$ . Now if (iv) does not hold there must exist a sequence  $\{A^i, B^i, C^i, D^i, a^i, c^i\}$ ,  $i = 1, 2, \dots$ , converging to  $(A, B, C, D, a, c)$  such that for  $i = 1, 2, \dots$ ,

$$A^i x + B^i y = a^i, C^i x + D^i y > c^i, x > 0$$

has no solution  $(x, y)$ . By Motzkin's theorem this is equivalent to

$$0 \neq \begin{pmatrix} -A^iT_u - C^iT_v \\ a^iT_u + c^iT_v \\ v \end{pmatrix} \geq 0, \quad B^iT_u + D^iT_v = 0$$

having a solution  $(u^i, v^i)$  for  $i = 1, 2, \dots$ . Letting  $\bar{u}^i = \frac{u^i}{\|u^i, v^i\|}$ ,

$\bar{v}^i = \frac{v^i}{\|u^i, v^i\|}$ , it follows by the Bolzano-Weierstrass theorem that there exists an accumulation point  $(\bar{u}, \bar{v})$  of  $\{(\bar{u}^i, \bar{v}^i)\}$  satisfying

$$\begin{pmatrix} -A^T \bar{u} - C^T \bar{v} \\ a^T \bar{u} + c^T \bar{v} \\ \bar{v} \end{pmatrix} \geq 0, \quad B^T \bar{u} + D^T \bar{v} = 0, \quad \|\bar{u}, \bar{v}\| = 1.$$

If  $\bar{v} \neq 0$  we contradict (iii'). If  $\bar{v} = 0$  and  $0 \neq -A^T \bar{u} \geq 0$ , then

$a^T \bar{u} \geq 0$  and  $B^T \bar{u} = 0$ , and again we contradict (iii'). Finally if  $\bar{v} = 0$  and  $-A^T \bar{u} = 0$ , then  $B^T \bar{u} = 0$ , and  $\|\bar{u}\| = 1$  which contradicts the linear independence of the rows of  $[A \ B]$ .

(iv)  $\Rightarrow$  (iii): Set  $\bar{A} = A$ ,  $\bar{B} = B$ ,  $\bar{C} = C$ ,  $\bar{D} = D$ ,  $\bar{a} = a$  and  $\bar{c} = c$ .

(iii)  $\Rightarrow$  (i): Let  $(b, d)$  be in  $R^{m_1+m_2}$ . Because the rows of  $[A \ B]$

are linearly independent there exists  $(\tilde{x}, \tilde{y})$  such that  $A\tilde{x} + B\tilde{y} = b$ . Let

$(\bar{x}, \bar{y})$  satisfy  $A\bar{x} + B\bar{y} = a$ ,  $C\bar{x} + D\bar{y} > c$ ,  $\bar{x} > 0$ . Then for sufficiently large positive  $\lambda$  we have  $\lambda \geq 1$  and

$$A(\tilde{x} + \lambda \bar{x}) + B(\tilde{y} + \lambda \bar{y}) = b + \lambda a, \quad C(\tilde{x} + \lambda \bar{x}) + D(\tilde{y} + \lambda \bar{y}) > d + \lambda c, \quad \tilde{x} + \lambda \bar{x} > 0.$$

Hence dividing by  $\lambda$  and defining  $\gamma = \frac{1}{\lambda}$ ,  $x = \frac{\tilde{x} + \lambda \bar{x}}{\lambda}$ ,  $y = \frac{\tilde{y} + \lambda \bar{y}}{\lambda}$  we obtain (i).

(i')  $\Rightarrow$  (vi'): Take  $g = 0$ ,  $h = 0$  and  $\alpha = 0$  in (vi') then by (i')

$$S(0, 0, 0) = \{0\}.$$

(vi')  $\Rightarrow$  (iii'): Suppose not, then either the rows of  $[A \ B]$  are linearly dependent or there exist  $(\tilde{u}, \tilde{v})$  satisfying

$$0 \neq \begin{pmatrix} -A^T \tilde{u} - C^T \tilde{v} \\ a^T \tilde{u} + c^T \tilde{v} \\ \tilde{v} \end{pmatrix} \geq 0, \quad B^T \tilde{u} + D^T \tilde{v} = 0.$$

In the former case there exists a  $\bar{u} \neq 0$  such that  $A^T \bar{u} = 0$ ,  $B^T \bar{u} = 0$  and  $a^T \bar{u} \geq 0$ , and hence the set  $S(g, h, \alpha)$  is empty or unbounded for each  $(g, h, \alpha)$  in  $R^{n_1+n_2+1}$  because  $S(g, h, \alpha) + \lambda(\bar{u}, 0) \subset S(g, h, \alpha)$  for  $\lambda \geq 0$ . Similarly in the latter case the set  $S(g, h, \alpha)$  is empty or unbounded for each  $(g, h, \alpha)$  in  $R^{n_1+n_2+1}$  because  $S(g, h, \alpha) + \lambda(\tilde{u}, \tilde{v}) \subset S(g, h, \alpha)$  for  $\lambda \geq 0$ . Hence in both cases we contradict (vi').

(i')  $\Rightarrow$  (v'): Suppose, not, then for some  $(g, h, \alpha)$  in  $R^{n_1+n_2+1}$  the set  $S(g, h, \alpha)$  is nonempty and unbounded. Hence there exists a sequence  $\{(u^i, v^i)\}$ ,  $(u^i, v^i) \neq 0$ ,  $i = 1, 2, \dots$ , such that  $\|u^i, v^i\| \rightarrow \infty$  and for  $i = 1, 2, \dots$ ,

$$\frac{A^T u^i + C^T v^i}{\|u^i, v^i\|} \leq \frac{g}{\|u^i, v^i\|}, \quad \frac{B^T u^i + D^T v^i}{\|u^i, v^i\|} = \frac{h}{\|u^i, v^i\|},$$

$$\frac{a^T u^i + c^T v^i}{\|u^i, v^i\|} \geq \frac{\alpha}{\|u^i, v^i\|}, \quad \frac{v^i}{\|u^i, v^i\|} \geq 0.$$

Hence by the Bolzano-Weierstrass theorem there exists a  $(\bar{u}, \bar{v})$  such that

$$A^T \bar{u} + C^T \bar{v} \leq 0, \quad B^T \bar{u} + D^T \bar{v} = 0, \quad a^T \bar{u} + c^T \bar{v} \geq 0, \quad \bar{v} \geq 0, \quad \|\bar{u}, \bar{v}\| = 1$$

which contradicts (i').

(v')  $\Rightarrow$  (i'): If not, then there exist  $(\bar{u}, \bar{v})$  satisfying

$$A^T \bar{u} + C^T \bar{v} \leq 0, \quad B^T \bar{u} + D^T \bar{v} = 0, \quad a^T \bar{u} + c^T \bar{v} \geq 0, \quad \bar{v} \geq 0, \quad (\bar{u}, \bar{v}) \neq 0$$

and hence for  $(g, h, \alpha) = (0, 0, 0)$ ,  $S(0, 0, 0)$  is nonempty and unbounded because

$$\lambda(\bar{u}, \bar{v}) \subset S(0, 0, 0) \quad \text{for } \lambda \geq 0. \quad \text{This contradicts (v').}$$

(i')  $\Rightarrow$  (vii'): Suppose not, then for each  $\varepsilon > 0$  there exists  $(\tilde{a}, \tilde{c})$  such that  $\|(\tilde{a}, \tilde{c}) - (a, c)\| \leq \varepsilon$  and some  $(g, h)$  such that the linear program of (vii') is feasible and either has an unbounded optimal solution set or no optimal solution. Hence for each  $\varepsilon > 0$  there exists an  $\alpha$  ( $\alpha$  being the

maximum of the linear program of (vii') if it has a solution, else  $\alpha$  is any real number) and a sequence  $\{(u^i, v^i)\}$ ,  $(u^i, v^i) \neq 0$ ,  $i = 1, 2, \dots$ , such that  $\{\|u^i, v^i\|\} \rightarrow \infty$  for  $i = 1, 2, \dots$ , and

$$\frac{A^T u^i + C^T v^i}{\|u^i, v^i\|} \leq \frac{g}{\|u^i, v^i\|}, \quad \frac{B^T u^i + D^T v^i}{\|u^i, v^i\|} = \frac{h}{\|u^i, v^i\|}, \quad \frac{v^i}{\|u^i, v^i\|} \geq 0, \quad \frac{\tilde{a}^T u^i + \tilde{c}^T v^i}{\|u^i, v^i\|} \geq \frac{\alpha}{\|u^i, v^i\|}.$$

It follows by the Bolzano-Weierstrass theory that for each  $\varepsilon > 0$  there exists a  $(\tilde{u}, \tilde{v})$  such that

$$A^T \tilde{u} + C^T \tilde{v} \leq 0, \quad B^T \tilde{u} + D^T \tilde{v} = 0, \quad \tilde{v} \geq 0, \quad \tilde{a}^T \tilde{u} + \tilde{c}^T \tilde{v} \geq 0, \quad \|\tilde{u}, \tilde{v}\| = 1.$$

Now by letting  $\varepsilon$  approach zero and noting that  $\|\tilde{u}, \tilde{v}\| = \|\tilde{u}(\varepsilon), \tilde{v}(\varepsilon)\| = 1$  and  $\|(\tilde{a}, \tilde{c}) - (a, c)\| \leq \varepsilon$  we obtain once again by the Bolzano-Weierstrass theorem that  $\{(\tilde{u}(\varepsilon), \tilde{v}(\varepsilon))\} \rightarrow (\bar{u}, \bar{v})$  and that

$$A^T \bar{u} + C^T \bar{v} \leq 0, \quad B^T \bar{u} + D^T \bar{v} = 0, \quad \bar{v} \geq 0, \quad \bar{a}^T \bar{u} + \bar{c}^T \bar{v} \geq 0, \quad \|\bar{u}, \bar{v}\| = 1$$

which contradicts (i').

(vii)  $\Rightarrow$  (i): Evident. Take  $\gamma = \min\{\frac{\varepsilon}{\|b, d\|}, 1\}$ .

(viii')  $\Rightarrow$  (vi'): Take  $\tilde{a} = a$ ,  $\tilde{c} = c$  and define  $\alpha$  to equal the maximum of the linear program of (vii') which by assumption has a unique solution. The set  $S(g, h, \alpha)$  of (vi') now consists precisely of this unique point.

(i')  $\Rightarrow$  (viii'): From (i') we have that the origin is the unique solution of the linear program

$$\max_{u, v} \left\{ \begin{array}{l} \tilde{a}^T u + \tilde{c}^T v \\ \left| \begin{array}{l} A^T u + C^T v \leq 0 \\ B^T u + D^T v = 0 \\ v \geq 0 \end{array} \right. \end{array} \right\}$$

for  $\|(\tilde{a}, \tilde{c}) - (a, c)\| \leq \varepsilon$  and any  $\varepsilon > 0$  because its feasible region contains the origin only. Hence (viii') holds for  $(g, h) = (0, 0)$ .

□



We note in Theorem 1 above that condition (iv) is a reproduction of the openness condition (iii) with  $\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{a}, \bar{c}$  replacing  $A, B, C, D, a, c$  respectively. From this we can immediately draw the following replication result.

Theorem 2 (Replication theorem). If any of the alternatives of Theorem 1 hold, then all of them hold with  $\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{a}, \bar{c}$  replacing  $A, B, C, D, a, c$  respectively where

$$\max\{\|\bar{A}-A\|, \|\bar{B}-B\|, \|\bar{C}-C\|, \|\bar{D}-D\|, \|\bar{a}-a\|, \|\bar{c}-c\|\} \leq \delta \text{ for some } \delta > 0 .$$

Remark 1. The classical Gordan theorem of the alternative is the equivalence between the alternatives (i') and (iii) of Theorem 1 with all data suppressed except the matrix  $D$ .

Remark 2. Some classical existence, stability and perturbation results for linear systems of equations follow from Theorem 1 by suppressing everything except the matrix  $B$  and the vector  $a$ .

### 3. Some applications of the stable theorem of the alternative

Theorem 1 can be used to obtain some interesting characterizations of boundedness and uniqueness of linear programming solutions. In particular we will show (see the equivalence (ii)  $\Leftrightarrow$  (v) of Theorem 3 below) that the solution set of a linear program is bounded if and only if the linear program remains solvable for all arbitrary but sufficiently small perturbations of the objective function coefficients. It is interesting to contrast this with the uniqueness characterization (see the equivalence (ii)  $\Leftrightarrow$  (v) of Theorem 4 below) which states that a linear programming solution is unique if it remains a solution to all linear programs obtained by arbitrary but sufficiently small perturbations of the objective function coefficients.

We state and prove now a boundedness characterization theorem for linear programming. A special case of the equivalence below between (i) and (ii) follows from Goldman's results [5, Corollary 1B] and has been given by Williams [15, Theorem 3]. However, Williams' theorem is incorrect without the additional unstated assumption that the primal feasible region is nonempty in the first part of the theorem, and that the dual feasible region is nonempty in the second part. The equivalence below between (i) and (iii) can also be derived from [13, Theorem 3].

Theorem 3 (Boundedness of linear programming solutions)

Let  $(g, h) \in R^{n_1 + n_2}$  and let

$$S = \{(u, v) \mid A^T u + C^T v \leq g, B^T u + D^T v = h, v \geq 0\}$$

$$\bar{S}(a, c) = \{(\bar{u}, \bar{v}) \mid (\bar{u}, \bar{v}) \text{ solves: } \max_{u, v} a^T u + c^T v \text{ s.t. } (u, v) \in S\}.$$

The following are equivalent:

(i)  $S$  is nonempty and

$$A^T u + C^T v \leq 0, B^T u + D^T v = 0, a^T u + c^T v \geq 0, v \geq 0, (u, v) \neq 0$$

has no solution  $(u, v)$ .

(ii)  $\bar{S}(a, c)$  is nonempty and bounded.

(iii)  $S$  is nonempty and the following constraint qualification is satisfied:

The rows of  $[A \ B]$  are linearly independent, and

$$Ax + By = a, Cx + Dy > c, x > 0$$

has solution  $(x, y)$ .

(iv) There exists  $\epsilon > 0$  such that  $\bar{S}(\bar{a}, \bar{c})$  is nonempty and bounded for all

$(\bar{a}, \bar{c})$  such that  $\|(\bar{a}, \bar{c}) - (a, c)\| \leq \epsilon$ .

(v) There exists  $\epsilon > 0$  such that  $\bar{S}(\bar{a}, \bar{c})$  is nonempty for all  $(\bar{a}, \bar{c})$

such that  $\|(\bar{a}, \bar{c}) - (a, c)\| \leq \epsilon$ .

#### Proof

(i)  $\Rightarrow$  (iv): Follows from the implication (i')  $\Rightarrow$  (vii') of Theorem 1 with the extra condition that  $S \neq \emptyset$  imposed on both (i') and (vii').

(ii)  $\Rightarrow$  (i): Obviously  $S \neq \emptyset$ . If there exists  $(u, v)$  such that

$$A^T u + C^T v \leq 0, B^T u + D^T v = 0, a^T u + c^T v \geq 0, v \geq 0, (u, v) \neq 0$$

then for any  $(\bar{u}, \bar{v}) \in \bar{S}(a, c)$  we have  $(\bar{u} + \gamma u, \bar{v} + \gamma v) \in \bar{S}(a, c)$  for  $\gamma \geq 0$ ,

and hence  $\bar{S}(a, c)$  is unbounded which is a contradiction.

(i)  $\Rightarrow$  (ii): If (ii) does not hold then  $\bar{S}(a, c)$  is empty or unbounded. In either case, since  $S$  is nonempty, there exist  $\{(u^i, v^i)\}$ ,  $i = 1, 2, \dots$ , with  $\|u^i, v^i\| \rightarrow \infty$  such that

$$\frac{A^T u^i + C^T v^i}{\|u^i, v^i\|} \leq \frac{q}{\|u^i, v^i\|}, \frac{B^T u^i + D^T v^i}{\|u^i, v^i\|} = \frac{h}{\|u^i, v^i\|}, \frac{v^i}{\|u^i, v^i\|} \geq \eta, \frac{a^T u^i + c^T v^i}{\|u^i, v^i\|} \geq \frac{\beta}{\|u^i, v^i\|}$$

where  $\beta = \max_{(u,v) \in S} a^T u + c^T v$  if  $\bar{S}(a,c)$  is nonempty, while if  $\bar{S}(a,c)$  is empty then since  $S$  is nonempty,  $a^T u^1 + c^T v^1 = \infty$  and hence  $\beta$  can be any fixed real number. By the Bolzano-Weierstrass theorem then, there exists  $(u,v)$  satisfying the conditions of (i).

(i)  $\Leftrightarrow$  (iii): Follows from the equivalence (i')  $\Leftrightarrow$  (iii) of Theorem 1.

(iv)  $\Rightarrow$  (v): Obvious.

(v)  $\Rightarrow$  (i): Obviously  $S$  is nonempty. Since  $\bar{S}(\bar{a}, \bar{c})$  is nonempty for

$\|(\bar{a}, \bar{c}) - (a, c)\| \leq \epsilon$  it follows that by linear programming duality that

$Ax + By = \bar{a}, Cx + Dy \geq \bar{c}, x \geq 0$  has solution  $(x,y)$  for

$\|(\bar{a}, \bar{c}) - (a, c)\| \leq \epsilon$ . It follows by the equivalence (i)  $\Leftrightarrow$  (i') of Theorem

1 that (i) of this theorem holds. □

We turn now to characterizing uniqueness of linear programming solutions. In [9] uniqueness-characterizing theorems similar to Theorem 3 above were obtained by using theorems of the alternative subsumed by Theorem 1 above. We give below a slightly more general result than that of [9] with a considerably simpler proof.

Theorem 4 (Uniqueness of linear programming solution)

Let  $g, h, S$  and  $\bar{S}(a,c)$  be defined as in Theorem 3. Let

$(\bar{u}, \bar{v}) \in \bar{S}(a,c)$  and let

$$J = \{i | (A^T \bar{u} + C^T \bar{v})_i = g_i\}, H = \{i | \bar{v}_i = 0\}, \tilde{H} = \{i | \bar{v}_i > 0\}.$$

The following are equivalent:

(i)  $(A^T u + C^T v)_J \leq 0$ ,  $B^T u + D^T v = 0$ ,  $a^T u + c^T v \geq 0$ ,  $v_H \geq 0$ ,  $(u,v) \neq 0$  has no solution  $(u,v)$ .

(ii)  $\bar{S}(a,c)$  is a singleton.

(iii) The following constraint qualification is satisfied: The rows of

$$\begin{bmatrix} A_{\cdot J} & B \\ C_{\tilde{H}J} & D_{\tilde{H}} \end{bmatrix} \text{ are linearly independent and}$$

$$A_{\cdot J} x_J + B y = a, C_{\tilde{H}J} x_J + D_{\tilde{H}} y = c_{\tilde{H}}, C_{HJ} x_J + D_H y > c_H, x_J > 0$$

has solution  $(x_J, y)$ .

(iv) There exists  $\epsilon > 0$  such that  $(\bar{u}, \bar{v})$  is the only element in  $\bar{S}(\bar{a}, \bar{c})$  for all  $(\bar{a}, \bar{c})$  such that  $\|(\bar{a}, \bar{c}) - (a, c)\| \leq \epsilon$ .

(v) There exists  $\epsilon > 0$  such that  $(\bar{u}, \bar{v})$  is in  $\bar{S}(\bar{a}, \bar{c})$  for all  $(\bar{a}, \bar{c})$  such that  $\|(\bar{a}, \bar{c}) - (a, c)\| \leq \epsilon$ .

#### Proof

(i)  $\Rightarrow$  (ii): By the second order sufficient optimality conditions of nonlinear programming [7, Theorem 3.2].

(i)  $\Leftarrow$  (iii): By the equivalence (i')  $\Leftarrow$  (iii) of Theorem 1 above.

(iv)  $\Rightarrow$  (v): Obvious.

(ii)  $\Rightarrow$  (iv): Suppose not, then there exists a sequence  $\{(a^i, c^i)\}$  converging to  $(a, c)$  such that the linear programs  $\max_{(u,v) \in S} a^{iT} u + c^{iT} v$  have

solutions  $(u^i, v^i)$ ,  $i = 1, 2, \dots$ , (this follows by the implication (ii)  $\Rightarrow$

(v) of Theorem 3 above) which are distinct from  $(\bar{u}, \bar{v})$ . Hence the sequence

$\{(u^i, v^i)\}$  satisfies for  $i = 1, 2, \dots$ ,

$$a^{iT} (u^i - \bar{u}) + c^{iT} (v^i - \bar{v}) \geq 0, (A^T (u^i - \bar{u}) + C^T (v^i - \bar{v}))_J \leq 0,$$

$$B^T (u^i - \bar{u}) + D^T (v^i - \bar{v}) = 0, (v^i - \bar{v})_H \geq 0.$$

Dividing by  $\|u^i - \bar{u}, v^i - \bar{v}\|$  and using the Bolzano-Weierstrass theorem gives a  $(u, v)$  satisfying the conditions of (i) and hence for sufficiently small  $\lambda > 0$ ,  $(\bar{u} + \lambda u, \bar{v} + \lambda v)$  is in  $\bar{S}(a, c)$  which contradicts (ii).

(v)  $\Rightarrow$  (i): Suppose not, then there exists  $(\hat{u}, \hat{v})$  such that

$$(A^T \hat{u} + C^T \hat{v})_j \leq 0, B^T \hat{u} + D^T \hat{v} = 0, a^T \hat{u} + c^T \hat{v} \geq 0, \hat{v}_H \geq 0, (\hat{u}, \hat{v}) \neq 0.$$

Consider now the linear program  $\min_{(u, v) \in S} (a + \delta \hat{u})^T u + (c + \delta \hat{v})^T v$ . For all

sufficiently small  $\delta > 0$  we have that  $(\bar{u} + \delta \hat{u}, \bar{v} + \delta \hat{v}) \in S$  and

$$\begin{aligned} (a + \delta \hat{u})^T (\bar{u} + \delta \hat{u}) + (c + \delta \hat{v})^T (\bar{v} + \delta \hat{v}) \\ = (a + \delta \hat{u})^T \bar{u} + (c + \delta \hat{v})^T \bar{v} + \delta (a^T \hat{u} + c^T \hat{v}) + \delta^2 (\hat{u}^T \hat{u} + \hat{v}^T \hat{v}) \\ > (a + \delta \hat{u})^T \bar{u} + (c + \delta \hat{v})^T \bar{v}. \end{aligned}$$

This shows that for all  $\delta > 0$  sufficiently small,  $(\bar{u}, \bar{v})$  is not in  $\bar{S}(a + \delta \hat{u}, c + \delta \hat{v})$ , which contradicts (v).

□

It is interesting to note the similarities between the five conditions (i) to (v) of Theorem 3 and 4 and also to note the replication of the boundedness or uniqueness conditions of (ii) in the perturbed problem of (iv).

Finally we mention that an interesting practical application of the Theorem 1 has been made in the design and control of a heat exchanger network [10]. In particular the theorem is used to give conditions under which maximum energy recovery is possible in a heat exchanger network under a certain disturbance range.

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